MTH241 Fall 2024: Quiz 09

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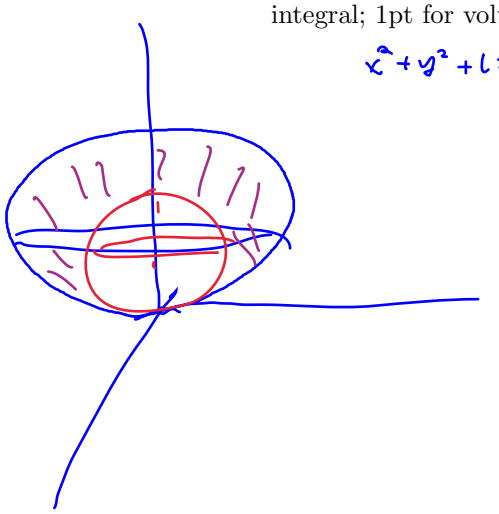
UID:

Closed book, no calculator, show your work clearly.

1. (5pt) Evaluate $\iiint_D 1dV$ to find the volume of D that is the solid region between the two spheres $x^2+y^2+z^2=2az$ and $x^2+y^2+z^2=az$. (Grading: 2pt working; 2pt for iterate integral; 1pt for volume)



$x^2+y^2+(z-a)^2=a^2$ $\qquad$ $x^2+y^2+(z-\frac{a}{2})^2=\frac{a^2}{4}$

$x^2+y^2+z^2=2az$

$\rho^2=2a\rho\cos(\varphi)$

$\Rightarrow \rho=2a\cos(\varphi)$

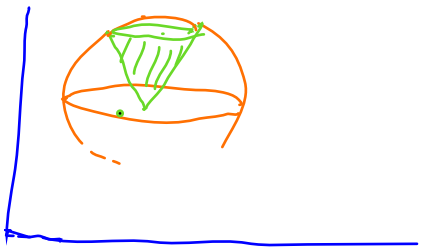
$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{2}}\int_{\rho=\text{inner sphere}}^{\rho=\text{outer sphere}} \rho^2\sin(\varphi)\, d\rho\, d\varphi\, d\theta$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{2}}\int_{\rho=a\cos(\varphi)}^{\rho=2a\cos(\varphi)} \rho^2\sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{7}{3}\sin(\varphi)\cdot a^3\cos^3(\varphi)\, d\varphi\, d\theta = \frac{7}{3}a^3\int_0^{2\pi}\int_0^{\pi/2}\sin(\varphi)\cos^3(\varphi)\, d\varphi\, d\theta$$

$$= -\frac{7}{12}a^3\int_0^{2\pi}\cos^4(\varphi)\Big|_0^{\pi/2}\, d\theta = \frac{7\pi}{6}a^3$$

2. (5pt) Find the surface area of the portion of the cone $z=\sqrt{x^2+y^2}$ that is inside the sphere $x^2+y^2+z^2=2$. Hint: Let $x=u\cos v, y=u\sin v$ what is the ranges of u and v? (Grading: 2pt working; 2pt for the area formula, 2pt for the correct area)



$z=\sqrt{x^2+y^2} \Rightarrow \boxed{\varphi=\frac{\pi}{4}}$

$\Rightarrow x=\rho\cos(\theta)\sin(\varphi) = \frac{\sqrt{2}}{2}\rho\cos(\theta)$

$y=\rho\sin(\theta)\sin(\varphi) = \frac{\sqrt{2}}{2}\rho\sin(\theta)$

$z=\rho\cos(\varphi) = \frac{\sqrt{2}}{2}\rho$

$$r(\rho,\theta)=\frac{\sqrt{2}}{2}\begin{bmatrix}\rho\cos(\theta)\\ \rho\sin(\theta)\\ \rho\end{bmatrix}$$

$r_\rho \times r_\theta = \rho\sin(\varphi)=\frac{\sqrt{2}}{2}\rho$

Overall: $S=\iint_R \|r_\rho \times r_\theta\|$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \frac{\sqrt{2}}{2}\rho\, d\rho\, d\theta$$

$$= \int_0^{2\pi} \frac{\sqrt{2}}{2}\cdot\frac{2}{4} = \boxed{\pi\sqrt{2}}$$

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